

## BARYCENTRIC EXTENSION OF GENERALIZED MATCHING

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In most studies of choice under concurrent schedules of reinforcement, two physically identical operanda are provided. In the “real world,” however, more than two choice alternatives are often available and biases are common. This paper describes a method for studying choices among an indefinite number of alternatives when large biases are present. Twenty rats were rewarded for choosing among five operanda with reinforcers scheduled probabilistically and concurrently. Large biases were generated by differences among the operanda: two were levers and three were pigeon keys. The results showed that when reinforcer frequencies were systematically varied, an extension of Baum’s (1974) Generalized Matching Model, referred to as the Barycentric Matching Model, provided an excellent description of the data, including individual bias values for each of the operanda and a single exponent indicating sensitivity to reinforcer ratios.

*Key words:* bias, barycentric analyses, concurrent schedules of reinforcement, probabilistic reinforcement, more than two choices, generalized matching, rats

The study of choices under concurrent schedules of reinforcement has been dominated by attempts to formulate models of behavior that describe and predict responding, whether at the general “molar” lever or at a response-by-response “local” level. The measure of a model is the extent to which it accounts for variance in existing data and how well it anticipates future behavior under novel schedules unfamiliar to the subject. One of the best-established models is Baum’s *Generalized Matching Law* (1974):

$$\frac{B_X}{B_Y} = k \left( \frac{R_X}{R_Y} \right)^s \quad (1)$$

The Generalized Matching Law describes, at a molar level, the relationship between ratios of two responses,  $B_X$  and  $B_Y$ , and two reinforcer frequencies,  $R_X$  and  $R_Y$ , across operanda  $X$  and  $Y$ . This description of the relationship between responses and reinforcers is accomplished with two parameters: sensitivity ( $s$ ) and bias ( $k$ ).

Sensitivity indicates how response ratios change with the reinforcer ratios. When response ratios exactly match reinforcer ratios,  $s = 1.0$ . When response ratios show a smaller

shift in ratio than do the reinforcer ratios,  $s < 1.0$ . For example, if reinforcers are awarded with a 9 : 1 ratio in favor of  $X$ , but the subject responds at a 3 : 1 ratio in favor of  $X$ , then  $s = 0.5$  (because  $9^{0.5} = 3$ ) and we would say the subject “undermatched.” If  $s > 1.0$ , then response ratios change in a more extreme fashion than the reinforcer ratios and the organism is said to “overmatch.” For example, if a 9 : 1 reward ratio elicits a 27 : 1 response ratio, then  $s = 1.5$  (because  $9^{1.5} = 27$ ).

The  $k$  parameter describes biases toward  $X$  (if  $k > 1.0$ ) or away from  $X$  (if  $k < 1.0$ ) relative to  $Y$ . Such biases appear to be independent of the reinforcer frequencies, and as such can be considered inherent preferences. Thus, if  $k = 2.0$ , we can interpret that as indicating that the subject preferred the  $X$  alternative twice as much as  $Y$ , and this preference did not depend upon the relative frequencies of reinforcement.

Most studies of responding under concurrent schedules have focused on two-choice situations, which is to say, those in which responses to two operanda were reinforced. A handful of exceptions exist in which more than two physically separate options were provided (e.g. Aparicio & Cabrera, 2001; Elsmore & McBride, 1994), and a few other studies simulated choices among more than two operanda by concatenating responses into sequences (e.g. Schneider & Davison, 2005), but the central focus has been on two-alternative scenarios.

It is also the case that efforts have been made in most experiments to “equalize” the operanda—to remove bias—such that values

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for the  $k$  parameter generally approximate 1.0. Attempts to minimize extraneous operandum variables date back to early concurrent schedule research (Findley, 1958) and operanda equalization remains a concern in modern experimental designs. While a few studies have compared topographically distinct operanda where biases might be expected (e.g. Davison & Ferguson, 1978; Sumpter, Temple, & Foster, 1998), most have used pairs of topographically identical operanda.

The focus both on two operanda and on equalized operanda stems from the reasonable goals of isolating experimental variables and mitigating confounding variables or interactions. Nevertheless, these conventions call into question the external validity of many conclusions in this field. After all, real-world schedules of reinforcement rarely provide only two possibilities, and those options are almost never identical in type, topography, or inherent value. Extending the experimental paradigm to a multitude of choice options where biases would be expected may supplement earlier studies in an important way by testing the generality of their conclusions and implications.

We therefore describe responding by rats when each of five operanda provided reinforcers according to concurrently operating schedules. Some of these operanda differed from others in terms of required response topographies and distances from the food magazine, and these were expected to differentially bias choices.

A difficulty arises, however, when applying traditional matching analyses to more than two simultaneously available operanda. The Generalized Matching Law, as given by Equation 1, does not by itself provide the means to fully describe behavior on five simultaneous operanda. The main problem is that, as will be seen, the value of the bias parameter  $k$  is generally considered to compare only two operanda, without a clear way of including additional alternatives. If we consider the implications of the equation, however, a way to model behavior on any number of operanda emerges. To reach this model, we must first explore the bias parameter,  $k$ .

#### *Elaborating On Bias*

Baum's parameter  $k$  has been understood to represent a relative value that compares two operanda. In the example given above, when  $k = 2.0$ , this implies a two-to-one preference of

operandum  $X$  compared to operandum  $Y$ . Thus,  $k$  is implicitly a ratio between two values. In making this point, Baum (1974) explicitly presents matching in the following terms:

$$\frac{B_X}{B_Y} = \left( \frac{k_X}{k_Y} \right) \cdot \left( \frac{R_X}{R_Y} \right)^s \quad (2)$$

Here, the  $k$  parameter in Equation 1 is expanded into a fraction composed of two separate measures of bias,  $k_X$  and  $k_Y$ , one for each operandum, with only the result of the division estimated in practice as the  $k$  parameter in Equation 1. Thus, the *absolute* values of  $k_X$  and  $k_Y$  are undefined. As will be shown below, however, conceptualizing the ratio explicitly in terms of two values is important when more than two operanda are involved.

If a third operandum  $Z$  is made available, the expanded form of the Generalized Matching Law would continue to allow the original ( $k_X/k_Y$ ) bias to be computed, again as a single value, while a separate analysis could be performed to compare operandum  $Y$  to operandum  $Z$ , resulting in the parameter ( $k_Y/k_Z$ ). One basic implication of the Generalized Matching Law is that because bias is a relative attribute (bias of one thing *versus* another) and because these two  $k$  fractions share a common parameter (both include  $k_Y$ ), then the three bias values, one per operandum, should be interrelated. Put a different way, if the Generalized Matching Law can be applied over an indefinite number of choice alternatives, then all of the bias parameters obtained through pairwise matching comparisons would be relative to one another in a consistent manner. Such consistency would be shown if ( $k_X/k_Z$ ), as estimated empirically using Equation 2, approximately equaled the cross-multiplication of ( $k_X/k_Y$ )  $\cdot$  ( $k_Y/k_Z$ ). This property, which we refer to as the *transitivity of bias ratios*, is important because it means that estimates of bias, directly calculated from the data, should provide information about other bias ratios, e.g., if we observe ( $k_X/k_Y$ ) and ( $k_Y/k_Z$ ), we will be able therefore to estimate ( $k_X/k_Z$ ).

Furthermore, if the individual ratios are transitive, then bias can be described as a single multidimensional ratio that indicates the relative biases of all operanda, taken together. This can be seen by considering the following: The ratio ( $k_X : k_Y$ ) is the equivalent of the fraction ( $k_X/k_Y$ ), and although the concepts of *ratio* and *fraction* are commonly thought to be equivalent, a fraction is only one

kind of ratio. Formally, *ratio* describes any number of interrelated values. Thus, the fractions  $(k_X/k_Y)$ ,  $(k_Y/k_Z)$ , and  $(k_X/k_Z)$  can be related in a three-way ratio  $(k_X : k_Y : k_Z)$ . The Generalized Matching Law (as embodied by Equation 2) leaves unstated how to relate three or more values. A multidimensional ratio provides the possible solution.

A multidimensional ratio is by definition a *barycentric set of values*. Barycentric values represent the coordinates of a point in space, determined through the use of a coordinate system in which the values must add up to an arbitrary constant (Bogomolny, 2008). Thus, within a barycentric system, any change to an individual value necessarily changes the other values so that all ratios remain fixed. The important consequence for our problem is that barycentric coordinate systems can represent relationships among any number of values. A simple example of a barycentric system is  $x + y + z = T$  (where  $T$  represents the scale, and can be any positive number). In the barycentric case, the scale of the equation is arbitrary (we could easily set  $T$  to 10, or 0.1, or any other positive number) because changing the scale has no effect on the *interrelations* among  $x$ ,  $y$ , and  $z$ . In other words, whether  $x + y + z = 3$  or  $10 \cdot (x + y + z) = 30$ , the relationship between  $x$  and  $y$  (that is, the ratio  $x : y$ ) remains unchanged. An added benefit of analyses using barycentric terms is the identification of the *scale*, as defined by the  $T$  parameter.

To visualize how barycentric values work, consider the following example:  $x + y + z = 3$ . A three-alternative barycentric system can readily be presented as coordinates within a triangle<sup>1</sup>, shown in Figure 1. Instead of perpendicular axes, as in a traditional coordinate system, each axis runs from the edge of the triangle to the opposing vertex. If we set any single barycentric value equal to zero, the resulting

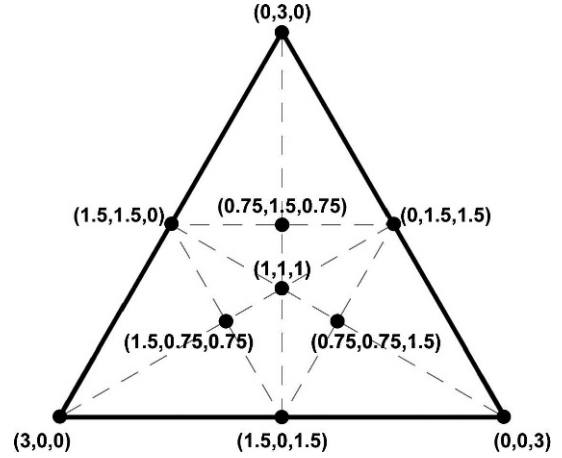


Fig. 1. Sample three-parameter barycentric coordinate system, where  $x + y + z = 3$ . When used to represent bias, the *barycenter* (at 1,1,1 in this case) represents the circumstance where the alternatives are equal to one another.

point must lie on the edge of the triangle. If we set two of the values equal to zero, the resulting point lies at the vertex of the triangle.

The *barycenter* of the triangle is the point at which all barycentric values are equal to one another. In Figure 1, that point is located at (1,1,1). For any other point in the system, however, the relations between the barycentric values can be readily understood as simple ratios. Consider the point (0.75, 1.5, 0.75). In this case, we can say that  $x$  and  $z$  had equal value, but both  $x$  and  $z$  were only half as large as  $y$ . Thus, the point (0.75, 1.5, 0.75) corresponds to a (1 : 2 : 1) relationship.

If bias is understood as a multidimensional ratio with barycentric properties, an arbitrary number of biases can be described and related, not merely the two values that a fraction permits. This way of representing bias, to be referred to as *barycentric bias*, has the properties we are accustomed to seeing in a bias ratio: a set of non-negative numbers whose relative values are consistent regardless of the scale of their absolute values.

One of the central objectives of this paper is to test the hypothesis that bias has barycentric properties. To test this hypothesis, we ascertain whether the pairwise bias estimates generated by Equation 2 display transitivity. For example, the closer  $(k_X/k_Y)$  is estimated by  $(k_X/k_Z) \cdot (k_Z/k_Y)$ , the more strongly transitivity will be confirmed. That confirmation, in turn, streng-

<sup>1</sup> In practice, barycentric values map to a point within a *simplex*. The most basic relevant example is a 1-simplex, which takes the form of a line segment divided into two portions by a point. A traditional bias ratio can thus describe the relative lengths of those portions, and is thus defined as the position of the point. A 2-simplex is a triangle, and corresponds to the example presented in Figure 1. A 3-simplex (or tetrahedron) can be used to describe a four-alternative set of barycentric values, which correspond to a point in three-dimensional space. Generalizing outward, the barycentric values for  $n$  alternatives represent a point within a  $(n-1)$ -simplex that can be mapped in  $(n-1)$  dimensions.

thens the proposal that we can describe bias as a barycentric system, with one bias parameter per operandum.

### *Simplifying Sensitivity*

Even if we can represent bias as having a barycentric character, it is not obvious from Equation 2 how to approach the sensitivity parameter  $s$ . Using Equation 2 to compare operand  $X$  and  $Y$  with one regression and  $X$  and  $Z$  with another regression might produce two different  $s$  parameters. If they differ, should that difference be viewed merely as the result of experimental error, or as an empirically meaningful effect? In trying to create a complete model of behavior, resolving this question is important.

The most comprehensive examination of sensitivity in cases where more than two operands are available was made by Schneider and Davison (2005). They present compelling evidence that a single common sensitivity value across all operands adequately describes matching behavior. If the multiple sensitivities derived from traditional matching analyses such as Equation 2 can be resolved into a single parameter, then we should in principle be able to describe matching behavior across an arbitrary number of operands with a single sensitivity value, plus one barycentric bias value associated with each operandum.

Thus, the second objective of this paper is to test whether a model with a single sensitivity parameter is sufficient to describe choices under concurrent schedules.

### *The Barycentric Matching Model*

The two hypotheses presented above (namely, that bias is barycentric and that a single sensitivity parameter suffices) lead to a model of performance that we will refer to as the *Barycentric Matching Model*. The basic premise of the Barycentric Matching Model is seen in Equation 3:

$$\frac{k_X \bullet (R_X)^s}{B_X} = \frac{k_Y \bullet (R_Y)^s}{B_Y} \quad (3)$$

This equation is arithmetically identical to Equation 2, but puts all values associated with a particular operandum on one side of the equation. If we use barycentric bias values (henceforth denoted as  $bk$ , with one  $bk_W$  per response alternative  $W$ ), as well as a single sensitivity parameter  $s$ , we can extend the

equation to any number of operands as shown in Equation 4, which defines the Barycentric Matching Model:

$$\begin{aligned} \frac{bk_A \bullet (R_A)^s}{B_A} &= \frac{bk_B \bullet (R_B)^s}{B_B} = \frac{bk_C \bullet (R_C)^s}{B_C} = \dots \\ &= \frac{bk_N \bullet (R_N)^s}{B_N} \end{aligned} \quad (4)$$

given that

$$(bk_A + bk_B + bk_C + \dots + bk_N = T)$$

Again,  $R_W$  here refers to reinforcer frequency on operandum  $W$ ,  $B_W$  refers to responses on operandum  $W$ ,  $s$  refers to sensitivity, and each  $bk_W$  refers to the barycentric bias associated with one operandum  $W$ . The  $T$  parameter is new to matching models, and is defined by the user as the scaling factor for the barycentric bias. We will use the convention throughout this paper that  $T$  will equal the number of available response alternatives. In our experiment, for example, we set  $T = 5$  because we utilized five operands. Clearly reporting the value for  $T$  is important because using a common scale facilitates the comparison of barycentric bias values across subjects or experiments. To this end, we will use  $bk_W$  parameters when we refer to bias parameters scaled to a fixed and explicit  $T$ , and  $k_W$  parameters to refer to bias parameters that have not been scaled in such a way.

Below, we present an experiment that tested the two central hypotheses of the Barycentric Matching Model: the barycentricity of bias parameters and the sufficiency of a single sensitivity parameter. The results of our analyses suggest that the Barycentric Matching Model can describe matching behavior effectively for an arbitrary number of discrete operands, even when those operands differ from one another.

## METHOD

### *Subjects*

Twenty 35-week-old male Long-Evans rats had been trained to respond on all operands but were otherwise experimentally naïve.

### *Apparatus*

The operant chambers measured 28 cm wide, 27 cm deep, and 29 cm tall, with a wire

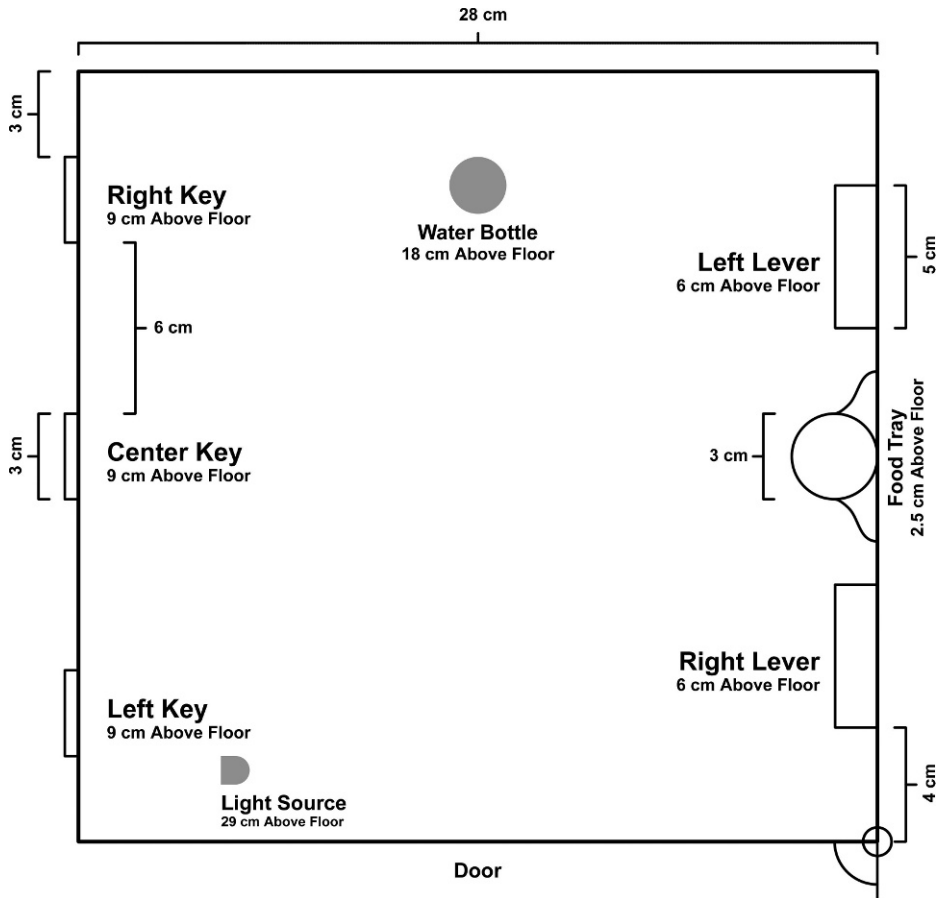


Fig. 2. Layout of the operant chambers, with the five operanda (Left, Center, and Right Keys; and Left and Right Levers), and other important parts of the apparatus.

mesh floor. Figure 2 shows the layout in the chambers. Five operanda were present in the chambers: three inset keys (3 cm diameter, 9 cm above the floor) on the left wall and two levers (5 cm wide, 1.5 cm deep, 1 cm thick, 6 cm above the floor) extending from the right wall. Reinforcers were delivered to a tray (3 cm diameter, 2.5 cm above the floor) on the right wall, located between the levers. Water was continuously available from a bottle 18 cm above the chamber floor and 3 cm forward from the back wall. The front wall of the chamber was a door. The front and back walls and the ceiling of the chamber were all composed of clear Plexiglas, while the left and right walls housing the operanda were made of aluminum. All mechanical components of the operant chambers were manufactured by Gerbrands. The chambers were contained

within sound-dampening enclosures, and a one-way mirror on the enclosure allowed the subjects to be viewed through the Plexiglas door of the chamber. Each chamber was controlled by an Apple eMac computer running Macintosh OS 9, applying a contingency programmed in TrueBasic.

#### *Procedure*

Reinforcers were concurrently scheduled on each of five operanda according to independently programmed random ratio schedules (Jensen & Neuringer, 2008; Lau & Glimcher, 2005). Controlling reinforcement on each of the five operanda was a random number generator and a probability-of-reinforcer-setup parameter. Preceding every response (regardless of which operandum was chosen), all five random number generators were activated (or



Table 1  
Probabilities of Reinforcer Setup.

Phase	Left Key	Center Key	Right Key	Left Lever	Right Lever
A	.03	.03	.03	.03	.03
B	.02	.03	.01	.04	.05
C	.025	.015	.08	.01	.02
D	.0475	.0425	.01	.035	.015
E	.0075	.06	.03	.0125	.04

“fired”) to produce a floating point number between 0.0 and 1.0, and a reinforcer was scheduled on any operandum whose random number was less than the probability-of-reinforcer-setup parameter associated with that operandum. Scheduled (or “set up”) reinforcers were “held” indefinitely until a response was made to that operandum. After a reinforcer had been scheduled on a given operandum, the random number generator for that operandum ceased firing, but continued to fire on other operanda for which reinforcers had not yet set up.

Thus, reinforcer setups were driven by responses (regardless of which operandum was responded to) and once set up, a reinforcer was collected with the first response to that operandum. A reinforcer setting up on one operandum did not prevent the random number generators on other keys from firing, and thus it was possible for reinforcers to set up and await collection on multiple operanda simultaneously. This contingency is functionally similar to concurrent variable-interval (VI) schedules, except that it is response-driven rather than time-driven. No changeover delays (or other penalties for switching) were used in the contingency (see Jensen & Neuringer, 2008).

Each of five phases provided a different set of reinforcer probabilities across the five operanda as depicted in Table 1. Each subject experienced a particular phase for five consecutive sessions, with the order of the phases counterbalanced across subjects. Each session lasted 90 min.

ANALYTIC METHOD

A series of analytic steps was necessary to calculate the parameter values for the Barycentric Matching Model, this being done separately for each subject. We will illustrate the analytic procedures with data from Subject

1 in the just-described experiment and provide all subjects’ data and additional analyses in the Results section to follow. Data from the third, fourth, and fifth sessions in each phase were used for all analyses, providing 15 sessions from which to calculate the barycentric bias associated with each operandum. For brevity, we will refer to the operanda as LK (Left Key), CK (Center Key), RK (Right Key), LL (Left Lever), and RL (Right Lever). Again, for clarity, we will use  $k_W$  to indicate a bias parameter without an explicit comparative scale, such as pairwise ( $k_X/k_Y$ ) estimates. By contrast, we will use  $bk_W$  to indicate barycentric bias values scaled to the constant  $T = 5$ . As will be seen, the  $bk_W$  values are estimated using a combination of  $k_W$  values, which are in turn estimated directly from the data.

Computing Barycentric Bias

We will describe one of two methods for estimating barycentric biases, the *Pairwise Method*, which makes use of Equation 2 to derive estimates of the parameters in Equation 4. An alternative, the *SALT Method*, produced very similar results as did Pairwise, and it is described in Appendix A. The two methods are arithmetically similar, and yield similar results in most, but not all, cases. For an overview of which method is best suited for a particular dataset, see Appendix B.

The Pairwise Method begins with a logarithmic transformation of the traditional form of the Generalized Matching Law (i.e. our Equation 2) applied to all possible pairs of operanda, e.g., ( $k_{LL}/k_{RL}$ ), ( $k_{LK}/k_{CK}$ ), and so on. This transformation is shown in Equation 5.

$$\log\left(\frac{B_X}{B_Y}\right) = \log\left(\frac{k_X}{k_Y}\right) + s \bullet \log\left(\frac{R_X}{R_Y}\right) \tag{5}$$

The result was a series of power functions, linear on log-log coordinates. To calculate the bias parameter for each unique pair of operanda, using Subject 1’s data, we estimated the ( $k_X/k_Y$ ) values from the intercepts of linear regressions on these data. Given five operanda, there are 25 (or  $5^2$ ) possible pairings, where operanda  $X$  and  $Y$  could be any pairing, such as LK vs. LL or CK vs. RL. Five of the pairings are *identities* (such as  $X$  vs.  $X$ ) whose bias ratios necessarily equal 1.0. Additionally, ten of the pairings are *inversions* (such as  $X$  vs.  $Y$  as opposed to  $Y$  vs.  $X$ ), such that calculating the values for ( $k_X/k_Y$ ) allows us to

Table 2  
Values of  $k_X/k_Y$  for Subject 1.

		Denominator ( $k_y$ )				
		Left Key	Center Key	Right Key	Left Lever	Right Lever
Numerator ( $k_x$ )	Left Key	1.000	1.024	0.987	0.502	0.556
	Center Key	0.977	1.000	0.971	0.492	0.551
	Right Key	1.014	1.030	1.000	0.510	0.570
	Left Lever	1.991	2.034	1.960	1.000	1.106
	Right Lever	1.800	1.814	1.753	0.904	1.000

derive values for  $(k_Y/k_X)$  by simply taking the inverse of the acquired value. Disregarding identities and inversions, there remain ten unique ratios in our five-operandum design (that is to say,  $(n^2 - n)/2$ , where  $n = 5$  in this case). For Subject 1, the resulting  $(k_X/k_Y)$  ratio values for each pair of operands are presented in Table 2.

Each cell represents the value of the  $k_X/k_Y$  fraction (as estimated from Equation 5), with row placement representing the numerator used to calculate the ratio and column placement representing the denominator. For example, Row 1, Column 2 presents the ratio derived from comparing  $B_{LK}/B_{CK}$  to  $R_{LK}/R_{CK}$  and observing the power function relationship between the two. We will denote the comparison of these using Equation 5 using the form  $(^{LK}/_{CK})$  hereafter for brevity. Note that five cells in the table (dark gray with white text) are set to 1.000: these cells are the *identities* mentioned above. Note also that 10 cells (light gray) are the redundant *inversions* mentioned above: the ratio calculated from  $(^{LK}/_{CK})$  is mirrored by its inverse,  $(^{CK}/_{LK})$ . In filling out this table, inversions are simply  $1/k$  relative to each appropriate ratio  $k$ . For purposes of barycentric calculations, all cells should be filled.

The Barycentric Matching Model assumes that bias ratios are transitive. If, for example,  $(k_X/k_Y = 1.5)$  and  $(k_Y/k_Z = 2.0)$ , then it should be the case that  $(k_X/k_Y \cdot k_Y/k_Z = k_X/k_Z = 3.0)$  through cross-multiplication. In practice, the values in Table 2 are merely estimates (from linear regressions) of the bias ratios, so some

degree of error can be expected. The efficacy of the Pairwise Method is largely determined by how strongly this transitivity assumption is born out by the actual data. As will be seen, the data offer strong support for transitivity.

To create a barycentric model, we rely on two observations. First, each value within a given column in Table 2 shares (or is scaled to) a common denominator. Because  $(^{LK}/_{CK})$  employs the same denominators as  $(^{RK}/_{CK})$ , we can compare LK to RK directly. Put differently, the ratio  $[(^{LK}/_{CK}) : (^{RK}/_{CK})]$  should equal the ratio (LK : RK). Second, all values in a single column are scaled to a different denominator than the values in every other column. Thus, we cannot directly compare  $(^{LK}/_{CK})$  to  $(^{LK}/_{RK})$ , because the two are scaled to different denominators.

In practice, because each column shares a scaling factor, it is already a barycentric set of values. However, each column's scale differs from that of its neighbors. Furthermore, each column is an estimate of the *same* set of barycentric values. This relationship can be demonstrated visually. Figure 3 (left), presents Table 2 as a series of lines: each line represents a column, and each shaded subsection of that line represents a single cell. The size of each subsection indicates the value of that cell. Note, for example, that because Column 1 generally has larger values than Column 4, the resulting line is longer. However, the absolute length of each line is not important: we are interested only in the ratios between subsections. Note, for example, that the darkest

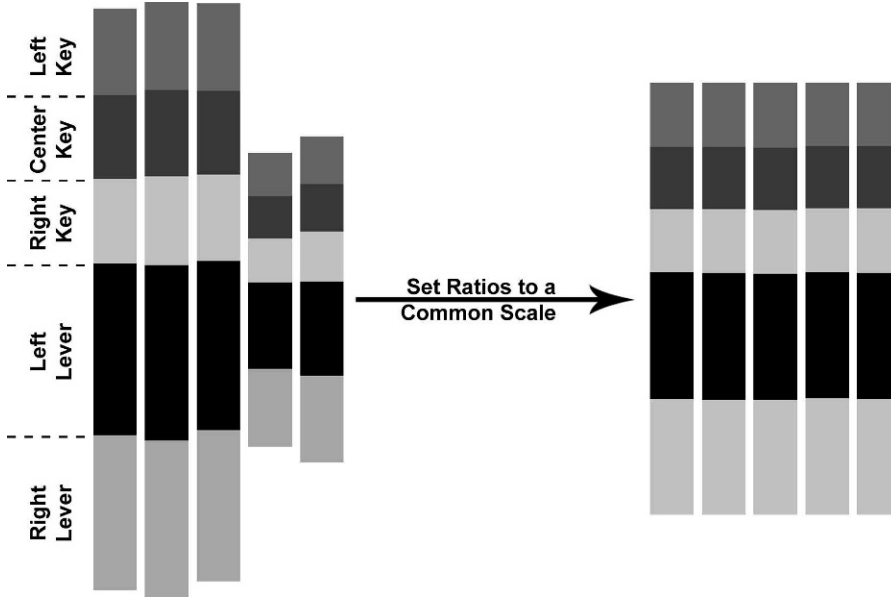


Fig. 3. Pictorial version of Subject 1's pairwise bias parameters (on the left and in Table 2) and the resulting barycentric bias when the scales are harmonized (on the right). Each bar represents one column in the tables, and each subdivision indicates one cell in that column. The height of each subsection matches the value of that cell.

subsection in each line (associated with the LL) is the largest subsection of each line regardless of that line's overall length.

To compare across columns directly, we must set them to the same scale, i.e. the denominators of all columns must be the same. Figure 3 (right) shows what this would look like by stretching and squashing the five lines to an equal height. As Figure 3 (right) demonstrates, these sets of ratios are actually similar to one another, even if they initially appear different in Figure 3 (left) due to differences in scale. The lines can be scaled in this way because each line in Figure 3 is an estimate of the same set of barycentric values, each initially scaled to a different constant  $T$ . Because the scale of a set of barycentric values is arbitrary, the differences between the lengths of the lines are not meaningful, and setting all of the values to a common value for  $T$  reveals their basic similarity.

The values in Table 2 can be brought into a common scale by taking the geometric mean of each row. For example, the geometric mean

$$\text{of first row is } \sqrt[5]{\frac{k_{LK}}{k_{LK}} \cdot \frac{k_{LK}}{k_{CK}} \cdot \frac{k_{LK}}{k_{RK}} \cdot \frac{k_{LK}}{k_{LL}} \cdot \frac{k_{LK}}{k_{RL}}} \text{ and}$$

$$\text{this reduces to } \frac{k_{LK}}{\sqrt[5]{k_{LK} \cdot k_{CK} \cdot k_{RK} \cdot k_{LL} \cdot k_{RL}}}.$$

With one geometric mean for each row, we now have five bias parameters (one for each operand), all scaled to the common denominator of  $\sqrt[5]{k_{LK} \cdot k_{CK} \cdot k_{RK} \cdot k_{LL} \cdot k_{RL}}$ . In order to conform to the convention that  $T = 5$ , we need only take the sum of our five bias estimates and multiply each parameter by  $\text{Sum}/5$ , such that the resulting parameters sum to  $T = 5$ . The result will be a series of  $bk_W$  parameters as defined in Equation 4.

In the case of Subject 1's data, the Pairwise Method produced bias estimates for the three keys and two levers (LK : CK : RK : LL : RL) which are, once scaled to  $T = 5$ , (0.738 : 0.724 : 0.749 : 1.468 : 1.321) respectively. We can immediately observe, for example, that while the Left Key and Center Key have a nearly one-to-one relative preference (more precisely, 0.738 : 0.724), the Left Key is almost exactly half as preferred as the Left Lever (0.738 : 1.468).

#### Assessing Sensitivity

To complete the analysis according to the Barycentric Matching Model, we calculate the value of the *sensitivity* parameter, a measure of how precisely subjects match ratios of responses to ratios of received reinforcers. Whether



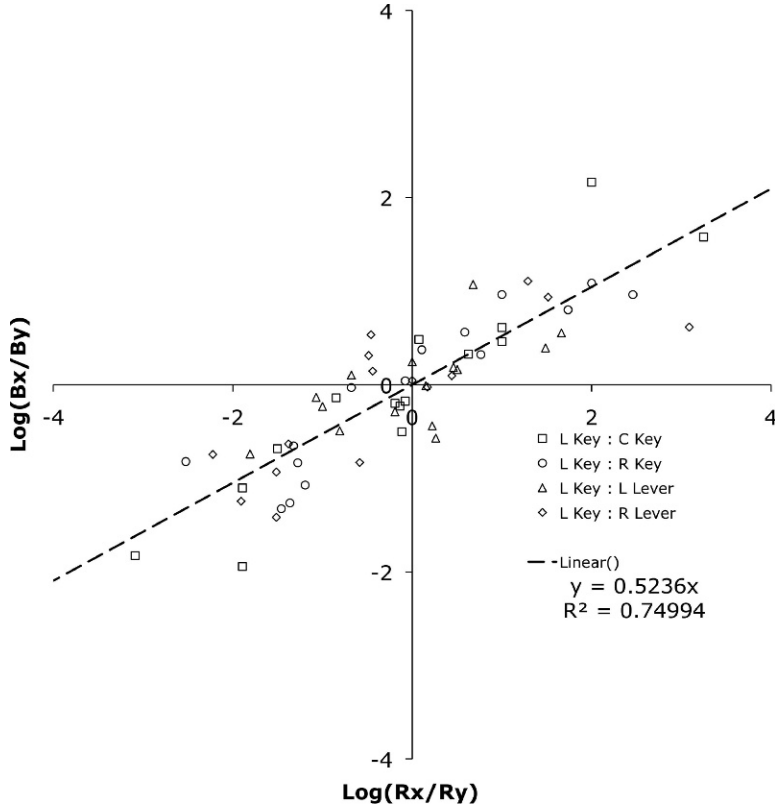


Fig. 4. Relationship between the bias-modified response ratios (*BRatios*), shown on the *y*-axis, and the ratios of obtained reinforcers, shown on the *x*-axis, for Subject 1, presented on logarithmic (base 2) coordinates. The Left Key is compared to each of the other operanda, as shown by a separate symbol, and the least-squared line is fitted to the entire set of points.

bias was calculated according to the Pairwise Method or the SALT Method described in Appendix A, the calculation for estimating sensitivity is the same. The prior estimation of barycentric bias enables us to factor out bias effects when computing sensitivity, with this being done separately for each pair of operanda as follows:

$$\log\left(\frac{B_X}{B_Y}\right) - \log\left(\frac{k_X}{k_Y}\right) = BRatio_{X:Y} \quad (6)$$

Instead of using a linear regression to compare the ratio of responses ( $B_X/B_Y$ ) with the ratio of reinforcers ( $R_X/R_Y$ ) (as in Equation 5), we plot a modified ratio of responses ( $BRatio_{X:Y}$ ) that has the ( $bk_X/bk_Y$ ) bias ratio factored out. The resulting function is expected to pass through the origin, and accordingly, no intercept

parameter is included in Equation 7:

$$BRatio_{X:Y} = s \cdot \log\left(\frac{R_X}{R_Y}\right) \quad (7)$$

The slope of the resulting function is our estimate of the *s* parameter.

As an example, Figure 4 plots *BRatios* as a function of reinforcer ratios in which the Left Key is compared to each of the other operanda (as indicated by the different symbols). Once the effects of bias are controlled for in this fashion, the four functions are similar to one another—so much so, in fact, that treating the data from all four comparisons as one dataset results in a highly representative best-fitting line (i.e. the function is well-described by a single sensitivity parameter). This result further supports the evidence for a single-

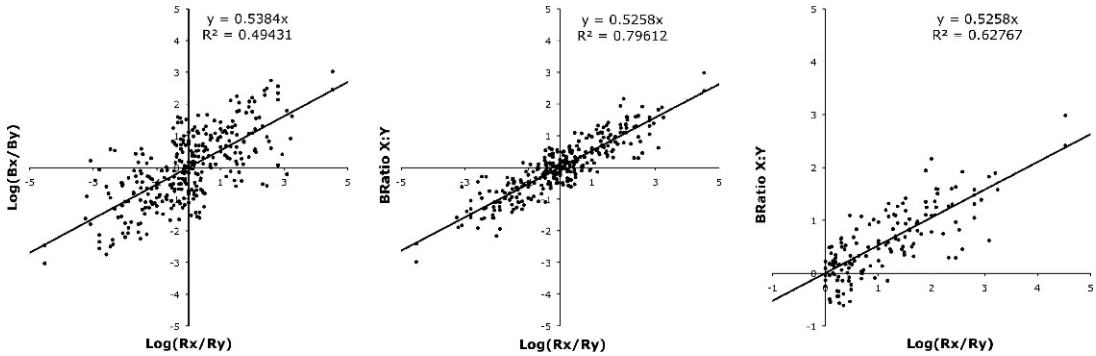


Fig. 5. The left graph shows the unmodified relationship between response ratios (on the y-axis) and reinforcer ratios (on the x-axis) for every pair of operanda. The center graph shows the relationship once response ratios have been modified to cancel out bias effect (*B Ratios*). The right graph shows the same data as the center, but excludes redundant ratios, all of which were located to the left of the origin. All graphs are presented on logarithmic coordinates (base 2).

sensitivity model reported by Schneider and Davison (2005).

Figure 5 applies the same reasoning as Figure 4, but does so exhaustively for all possible response ratios (excluding identities). Thus, for our experiment, this collapses 20 variations of Equation 7 into a single regression. Figure 5 (left) depicts unmodified comparisons of responses to reinforcers. By modifying each value on the y-axis to a *B Ratio* value instead (as in Equation 7), we substantially reduce the noise, resulting in Figure 5 (center). As is both visually and rationally obvious, however, Figure 5 (center) is actually a symmetrical function because it uses each response ratio twice. For example, the ratio of (LK : RL) is used as well as the ratio of (RL : LK). The most conservative way to circumvent this problem is to utilize points on a single side of the origin, as presented in Figure 5 (right). Because the intercept is constrained to the origin, the slopes in Figure 5 (center) and Figure 5 (right) are necessarily identical – removing the mirrored ratios serves mainly to change the  $R^2$  of the regression. We can estimate that  $s = 0.526$  for Subject 1 from the slope of this function.

We now have a complete description of Subject 1's behavior according to the Barycentric Matching Model, with five individual  $bk_W$  values and  $s = 0.526$ . We shall provide similar analyses for each subject, the goal being to define the biases for each of the five operanda and the sensitivities of responses to reinforcers.

RESULTS AND DISCUSSION

Figure 6 presents the arithmetic means of all rats' proportions of responses and received reinforcers during each session of the experiment, as well as the proportional probabilities of reinforcer setup. These represent the raw data used to compute the ratios employed by our subsequent analyses. Response proportions were calculated by dividing the number of responses to an operandum by the sum of responses to all operanda. Reinforcer proportions were calculated similarly. Proportions of responses and reinforcers are shown separately for each of the five operanda, one per box. Obtained reinforcer proportions were highly correlated with programmed reward proportions (correlation  $> .99$  for all operanda). Response allocations were also clearly related to obtained reinforcers, but responses showed less pronounced shifts than rewards in all cases; that is, subjects displayed undermatching. Additionally, signs of bias are suggested by these data. Relative responses to the Left Lever were consistently higher than relative reinforcers, for example, while the opposite was true for the Center Key.

Before we applied our analytic method to derive a Barycentric Matching Model, we tested the assumption of bias transitivity. Figure 7 (left) shows the relationship between the actual fitted intercept  $\text{Log}(k_X/k_Y)$  for every pair of operanda for every subject (using Equation 5), and the expected intercept, given an assumption of transitivity, calculated as

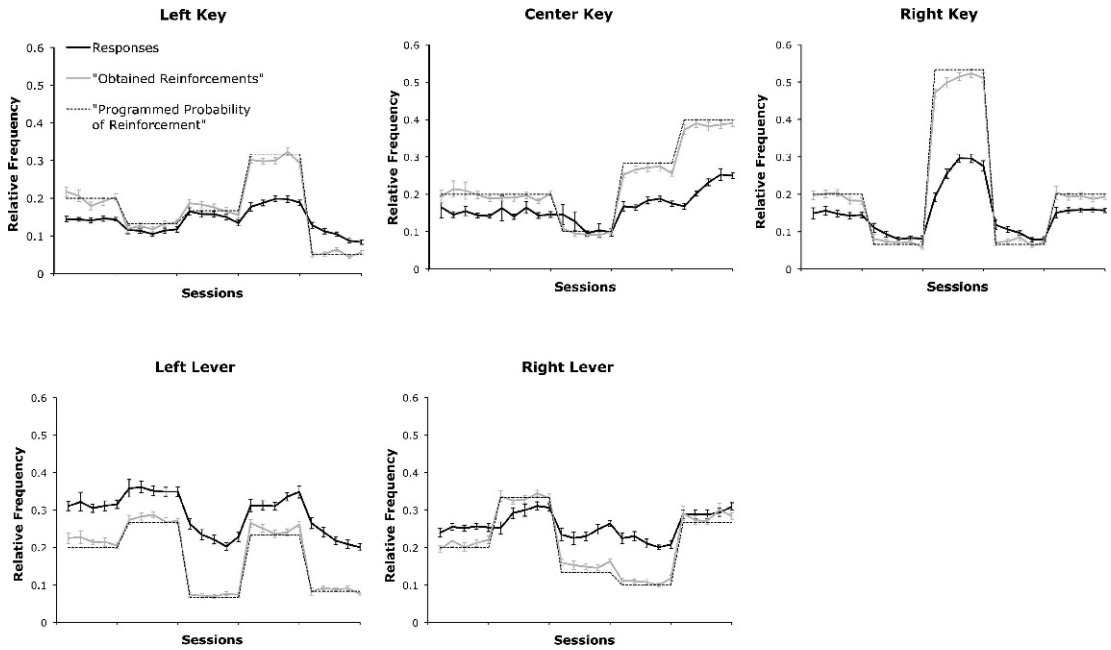


Fig. 6. The solid black lines show the average of all 20 subjects' relative rates of responding ( $B_X/[B_{LK}+B_{CK}+B_{RK}+B_{LL}+B_{RL}]$ ), with each operandum (namely, the Left Key (LK), Center Key (CK), Right Key (RK), Left Lever (LL) and Right Lever (RL)) in its own graph. The x-axis of each graph represents the 25 experimental sessions. Also shown are the analogous rates of obtained reinforcers (gray lines) and relative rates of programmed reinforcers (dotted lines). Error bars indicate one standard error.

$[\log(k_X/k_Z) + \log(k_Z/k_Y)]$ . There were three possible ways to calculate the expected intercepts, this done by cross-multiplications for every obtained ratio given by a response pair. For example, to calculate  $[\log(k_{LL}/k_Z) + \log(k_Z/k_{CK})]$ , operandum Z could be RK, LK, or RL. All three cross-multiplications are included in the figure. The figure shows a

close relationship between obtained and expected values, with a slope of 0.978 and an  $R^2$  of .979, and thereby strongly supports a claim of transitivity.

An important point to make here is that transitivity is assumed by the Barycentric Matching Model but not strictly determined by it. Figure 7 (center) and Figure 7 (right)

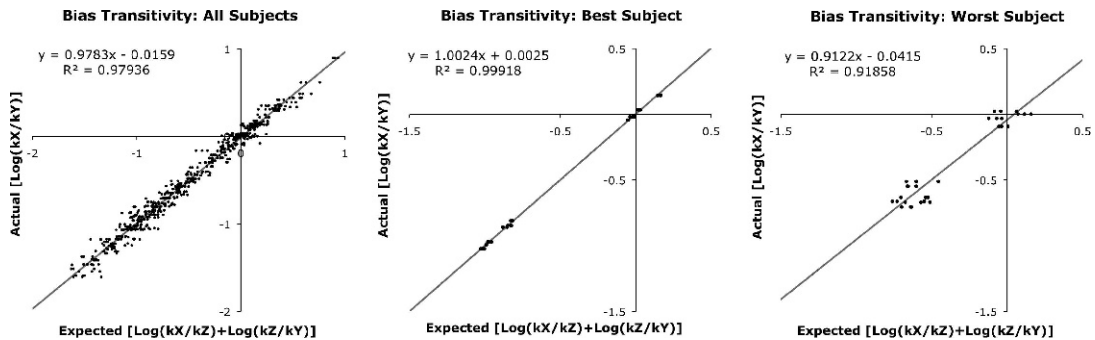


Fig. 7. These graphs indicate how well the assumption of bias transitivity is supported by the data. Plotted are the  $\log(k_X/k_Y)$  ratios obtained using the Generalized Matching Law (Equation 5), compared to the ratios predicted by transitivity (e.g.  $\log[k_X/k_Z] + \log[k_Z/k_Y]$ ). To the extent that the slopes approximate 1.0, the transitivity assumption is supported. The left graph shows data from all 20 subjects, combined into a single regression. The center graph shows data from the "best" subject (i.e. the animal that most strongly supports the transitivity assumption), and the right graph shows the data from the "worst" subject.

represent the subjects with the best and worst transitivity, respectively. As Figure 7 (right) demonstrates, individual points may deviate from the fit lines to some extent.

Given that the transitivity assumption was well met, a Barycentric Matching Model was constructed for each subject using data from the third, fourth, and fifth sessions from each of the five phases (15 sessions total) as described in the Analytic Method. Figure 8 presents each of the 20 rats' biases for each of the five operanda on a base 2 logarithmic scale<sup>2</sup>. In general, the levers were highly preferred and the keys dispreferred: a repeated-measures analysis of variance found a significant difference among biases,  $F(4,20) = 170.1$ ,  $p < .0001$ , and post-hoc paired  $t$ -test comparisons revealed significant differences between each of the keys on the one hand, and each of the levers on the other,  $t(19) > 13.3$  and  $p < .0001$  for all lever-to-key comparisons. Additionally, 18 out of 20 rats displayed a clear preference for the Left Lever over the Right Lever,  $t(19) = 4.42$ ,  $p < .0005$ . For the most part, subjects treated the three keys as equally preferred,  $t(19) < 1.6$ ,  $ns$  for all key-to-key comparisons, although 3 subjects displayed a slightly lower preference for the center key.

Factoring bias out, we then estimated each subject's sensitivity to reinforcer ratios. Figure 9 shows the distribution of estimated sensitivity values from the Barycentric Matching Model for each of the 20 subjects, as well as a kernel density estimation<sup>3</sup> of the distribution of those sensitivities. Overall, sensitivities were unimodally distributed, with a mean value of 0.4944 and median of 0.4939 and sensitivities

differed across a range of almost 2 to 1. We also calculated, for each rat's sensitivity value, a 99% confidence interval. Confidence intervals ranged from 0.029 to 0.064, with a mean of 0.044. Thus, the different sensitivities observed across subjects were considerably larger than could be explained by error alone. From these results we can conclude that, with respect to sensitivity to reinforcement, rats exposed to our five-operandum environment displayed undermatching and that individual differences were manifest.

We also tested the Barycentric Matching Model assumption that a single sensitivity value ( $s$ ) would suffice to describe choices between any two of the operanda, for example, when LK was compared to RK, LK to LL, etc. To calculate the barycentric biases, we performed 10 pairwise comparisons, each of which yielded two parameters, slope and intercept. For five operanda, there are 10 unique pairs, and thus 10 slopes and 10 intercepts for a total of 20 parameters. The question is whether utilizing 10 separate slopes yielded appreciably better descriptions than our single slope estimate. The Schwarz-Bayes Information Criterion (Schwarz, 1978), or "SIC", enabled us to compare this 20-parameter model to the 6-parameter Barycentric Matching Model. The SIC provides a relative score based on (a) the number of parameters in the model, (b) the residual sum of squares of the differences between the model and the observed data, and (c) the number of observations in the data set. The result is a number that enables comparison across models: the lower the score, the better the model. Thus, we compared the log ratios of responses to those predicted by the two models for all pairs of operanda. There were 10 pairs per session, across 15 sessions, for a total of 150 points. The SIC value associated with the Barycentric Matching Model was lower for all 20 subjects (mean difference = 48.17, standard error of differences = 2.20), and this difference was statistically significant according to a paired  $t$ -test,  $t > 21.9$ ,  $p < .0001$ . In order to disambiguate the benefits of barycentric bias from the benefits of a single slope, we also compared the Barycentric Matching Model to a model with 10 sensitivities but with the same five bias values used in the Barycentric model; that is, only the sensitivity parameters differed. The single-sensitivity model was still the better

<sup>2</sup> A logarithmic transformation is an appropriate means of equalizing variance for barycentric biases, but we feel that logarithms are inappropriate for reporting barycentric bias given that the equation for the Barycentric Matching Model does not use logarithms. We have included the untransformed parameters of the Barycentric Matching Model for each subject in Appendix C.

<sup>3</sup> Kernel density estimation is a nonparametric alternative to a histogram for presenting the density distribution of a series of values. In lay terms, a "kernel" (i.e. a small probability distribution) is placed at each position along the continuum of values, and those individual distributions are summed. The size of the kernels is determined by a "bandwidth" parameter, which is selected to minimize the asymptotic mean integrated squared error (or AMISE). In our case, a Gaussian kernel (i.e. a small normal distribution) with a bandwidth of 0.0364 was used. Kernel density estimation was popularized by Parzen (1962) and is discussed at length by Silverman (1986).

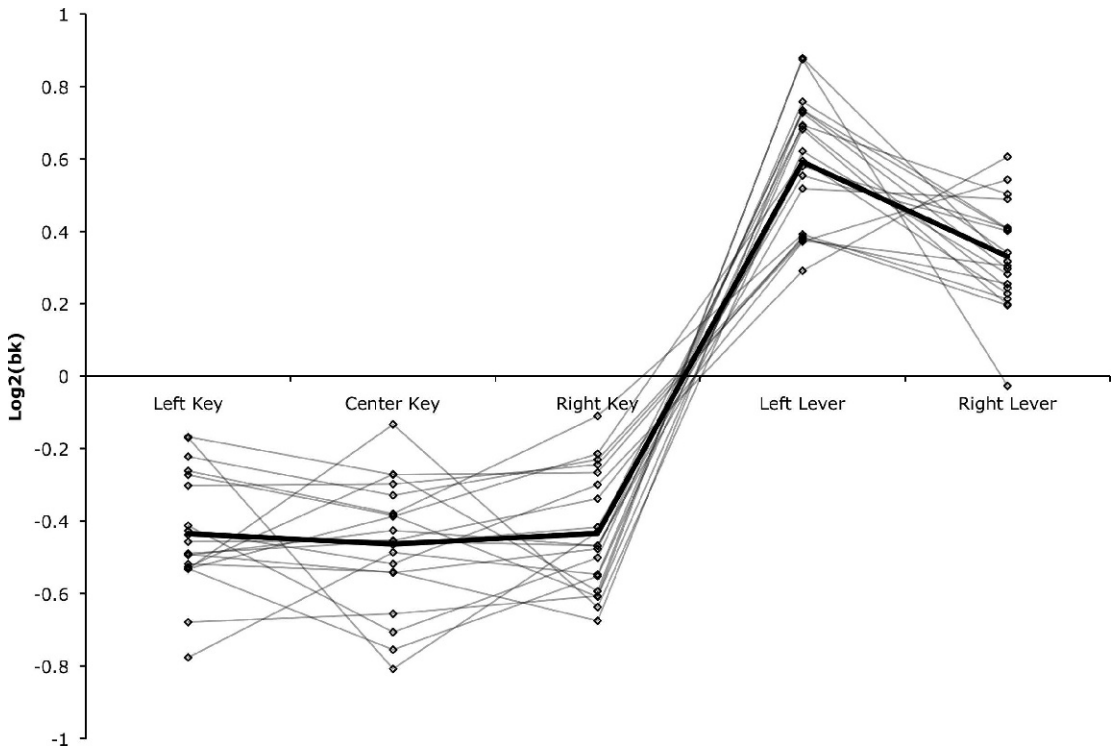


Fig. 8. Barycentric biases for each of the five operanda, logarithmically transformed (base 2). The bias values for each of the 20 rats are connected with lines. The dark line indicates the means across all subjects. Equal bias (i.e. an absence of preference) is indicated by the  $x$ -axis. Subjects can be said to show a bias against operanda with values below the axis and a bias for operanda above the axis.

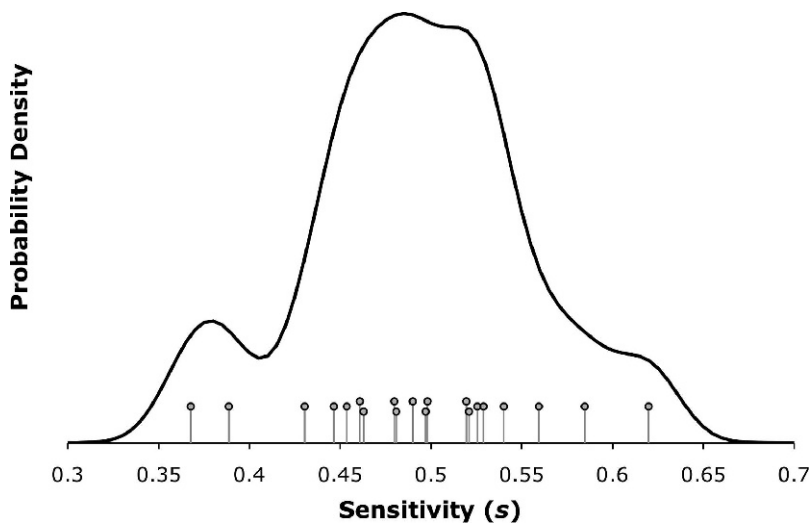


Fig. 9. Distribution of subjects' sensitivity parameters ( $s$ ), indicated as small gray circles, according to the Barycentric Matching Model. The height of the estimated probability density function indicates the relative probability that a subject will display the sensitivity indicated on the  $x$ -axis. The distribution was estimated using kernel density estimation with a Gaussian kernel and a bandwidth of 0.0364, selected to minimize the asymptotic mean integrated squared error (Silverman, 1986).



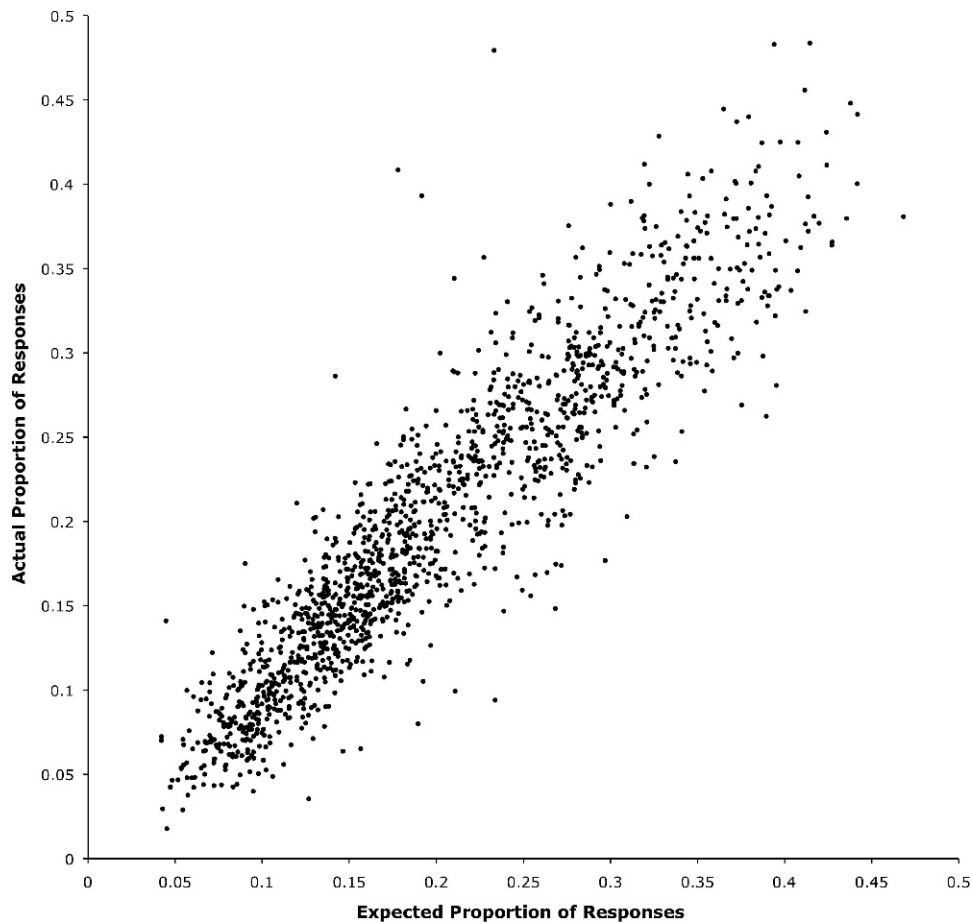


Fig. 10. Response proportions generated by each of the rats on all five operanda as a function of predicted response proportions. Predictions were based on each animal's Barycentric Matching Model. The predictive model was based on data from the terminal three sessions in each of the five phases. Across all operanda, actual and expected data were correlated with a coefficient of .9225.

model (mean difference = 24.2,  $t > 11.9$ ,  $p < .0001$ ). These comparisons therefore are consistent with the sufficiency, and possibly the advantage, of a single-slope model.

How well did the Barycentric Matching Model describe behavior? Figure 10 shows how well the parameter values estimated by the Barycentric Model describe the actual choice proportions generated by each of the rats. The figure shows the relationship between actual response proportions and those expected, based on the model, from all subjects on all operanda, in the third through fifth sessions of all phases (for a total of 1500 points). When we compared each individual subject's actual behavior to predictions

from the model, the median correlation was .9320, with the distribution of correlations skewed toward the higher values: 4 subjects' correlations were below a value of .9 and 7 subjects' correlations were above .94. In general, therefore, the Barycentric Matching Model provided a relatively accurate description of performances.

Use of a correlation on the data in Figure 10 is problematic, however, because the points are not normally distributed (nor are they on logarithmic coordinates). To supplement our comparison, we also performed Spearman's rank correlations to provide a nonparametric comparison of model predictions to subject performance. We performed a separate anal-

ysis for each subject, with the five relative frequencies of responses in all 15 sessions (for a total of 75 points). In all cases, the relationship between actual and expected response proportions were highly significant, with Spearman's  $\rho$  values ranging from .45 to .8,  $t(73) > 4.0$  and  $p < .00005$  for all subjects. The response proportions were not strictly independent of one another (as more responses to one operandum necessarily means fewer to another), so we also ran Spearman's rank correlations separately for each animal on each operandum, finding statistical significance in nearly every instance,  $t(13) > 0.48$  and  $p < .04$  for 95 out of the 100 comparisons. Thus, we conclude that the Barycentric Matching Model provided a good description of behavior.

### GENERAL DISCUSSION

Most studies of choice under concurrent schedules of reinforcement have focused on situations in which two alternatives are available (e.g., two keys, levers, or buttons, depending upon species) and in which inherent differences between the two operandum are minimized (e.g., differences in force necessary to operate them, distance from the reinforcer, and so forth). A commonly reported result of these experiments is that the ratios of emitted responses and collected reinforcers exhibit power-function relationships to one another. In the world outside of the lab, however, humans and other animals generally have more than two choice options at any given time. Foraging animals often have a wide variety of different areas to search. People can choose among many TV channels, many items on a menu, or many tasks around the house. Furthermore, those options are generally not subjectively equivalent. Differences in preference have a powerful effect outside the lab, whether they stem from the properties of the choice (e.g., relative quality, ease, or salience) or of the organism (e.g., handedness).

In the few reports of choice under concurrent schedules where more than two options (or operandum) are available, similar power function relationships have been reported. However, researchers have grappled with the challenge of how to analyze and describe data collected when more than two response alternatives are simultaneously available, a problem that has been especially acute with

respect to bias. With this in mind, we studied how 20 rats allocated their choices across five operandum when those options were designed to engender different levels of inherent bias (namely, two levers in close proximity to the reinforcer dispenser and three pigeon keys 25 cm from the dispenser). Our goal was to provide a succinct model that could describe choices in the presence of many options (two or more), even when the options potentially differed in terms of biases. We revised the Generalized Matching Law proposed by Baum (1974) based on the observation that the five operandum were related to one another as a multidimensional ratio. This barycentric property permitted us to compute one "barycentric bias" parameter for each operandum.

The 20 rats displayed highly consistent biases. All subjects markedly preferred the levers relative to the keys, and most subjects (18 of 20) favored the left lever over the right lever. The preference for levers over keys was expected given the close proximity of the levers to the reinforcer tray and to our informal observations that training lever presses was more rapid than training key presses. The preference of left relative to right lever is less easily explained. Recent informal observations in our lab suggested that the proximity of the water bottle to the Left Lever may have been responsible. Given the lack of a corresponding Right Key bias, we speculate that the Left Lever's position between the reinforcer delivery tray and the water may have been the cause.

Once these biases were derived from the barycentric analysis, sensitivity (i.e., the exponent of the power function) proved readily calculable. Consistent with other results in the literature, we found that a single sensitivity was sufficient to describe each animal's responses across all operandum. The resulting model, which consists of one bias parameter per operandum and one sensitivity parameter, is referred to as the Barycentric Matching Model. We applied the Schwarz-Bayes Information Criterion to compare the Barycentric Matching Model (with a single sensitivity parameter and five bias parameters) to the alternative where each pair of operandum, e.g., Left Lever and Right Key, had associated sensitivity and bias values, producing a twenty-parameter model. The Barycentric Matching Model was more parsimonious and was evaluated by the SIC statistic as the "better" model.

In contrast to the across-subject consistency of the bias results, there was considerable individual difference in the sensitivity values, yielding an approximately normally distributed spread, with the  $s$  parameter ranging from 0.368 to 0.620. Given that the sensitivity parameter's 99% confidence interval was no larger than 0.064 for any subject (and often much smaller), we conclude that the differences among subjects are too large to be dismissed as the result of experimental noise or error. The orderly distribution of  $s$  values is of interest because differences in sensitivity to reinforcement may be related to developmental, experiential, and genetic contributions to choice, as well as the effects of physiological and psychopharmacological manipulations. Such differences may also be correlated with other possible behavioral variables, such as ability to respond to delayed reinforcers or to engage in self-controlled choice responses. We do not at present know which aspects of our procedure (e.g., probabilistic reinforcers, high reinforcement frequency, more-than-two operanda, biased options) contributed to the distribution of sensitivity values, but the question warrants further study. A relatively complex environment, such as the one in the present studies, may be more likely than a simpler environment to distinguish sensitivity values.

The programming of reinforcers probabilistically, based on responses instead of passage of time, as under the more common VI schedules, deserves special comment. As in the concurrent VI VI case, where passage of time results in the possibility of reinforcers setting up on any of the available operanda, responses in our experiment served the same function, with a response to any of the five operanda probabilistically setting up a reinforcer on any of them. This procedure has only rarely been used in the study of choice (Jensen & Neuringer, 2008; Lau & Glimcher, 2005), but it may have advantages. In the concurrent VI VI case, when programmed reinforcers are very frequent, control over obtained reinforcement frequencies is lost, since almost any response results in reinforcement. Under our procedure, the different reinforcer probabilities were maintained even though these probabilities yielded much more frequent reinforcers than is commonly found in concurrent schedules. On average, every sixth response produced reinforcement, and this richness of feedback may have contributed to

the rapid acquisition displayed by subjects in Figure 6. This rapidity, in turn, allowed us to reliably estimate the parameters of matching behavior across five operanda (both in terms of operandum preferences and in terms of sensitivity to reinforcement contingencies) with only 5 sessions per phase for a total of 25 sessions in the entire experiment. In summary, concurrent probability schedules might enable rapid specification of choice performance. The employment of five (or more) separate choice options may also contribute because of the number of different reinforcement frequencies that can simultaneously be evaluated.

#### *SALT Estimates of Barycentric Bias*

Appendix A provides an alternative to the Pairwise method that was used to estimate barycentric bias values. This alternative is based on Natapoff's (1970) *Symmetric Approximation by Leading Term*, or SALT. The SALT equation (sometimes called the *Natapoff equation*) has been used in a number of studies, most notably Schneider and Davison (2005). Only minimal differences resulted from using the Pairwise Method described above and the alternative SALT Method for calculating barycentric bias in this study's data, but there exist situations in which one method would be preferred. Appendix B outlines the criteria for choosing between the two methods.

#### *Applying the Barycentric Matching Model to Other Experimental Data*

Schneider and Davison (2005) used matching models to analyze response sequences on two operanda concatenated in a way to create four choice options. Pigeons pecked two keys (Left and Right), and the "response unit" for the contingency was a two-response sequence, resulting in four possible sequences: LL, LR, RL, and RR. Schneider and Davison reported two important results: (1) Responses requiring stays (namely, LL and RR) were preferred over those requiring changeovers (LR and RL), and (2) a single sensitivity parameter was sufficient to describe behavior, despite there being six distinct possible "operant ratios" (such as LL : LR, RL : RR, etc.).

To demonstrate that a single sensitivity was adequate, Schneider and Davison (2005) utilized a bootstrapped optimization procedure to fit multiple variations of Equation 2

Table 3  
Sensitivity and Barycentric Bias in Schneider & Davison (2005).

Subject	Reported Results	Recalculated Results				
	Sensitivity	Sensitivity	$bk_{LL}$	$bk_{LR}$	$bk_{RL}$	$bk_{RR}$
111	0.82	0.822	1.320	0.629	0.699	1.352
112	0.53	0.526	1.621	0.661	0.517	1.200
113	0.84	0.842	0.972	0.983	0.828	1.217
114	0.65	0.652	1.179	0.750	0.788	1.283
115	0.73	0.728	1.039	1.045	0.600	1.315
116	0.65	0.650	1.085	0.818	0.823	1.274

simultaneously rather than employing an algebraic solution such as the one proposed in the present article. Furthermore, bias ratios were described in general terms (e.g. subjects “showed some biases favoring sequences with no changeovers,” p. 55) but the magnitudes of the biases were not discussed.

We applied the Barycentric Matching Model and found nearly identical parameter values, providing strong support for the authors’ claim of a single sensitivity. Furthermore, our analysis of the data in Schneider and Davison’s (2005) Appendix A provides additional quantitative evidence concerning the nature of the repetition bias.

As Table 3 demonstrates, the sensitivities computed algebraically with the Barycentric Model for each subject are (given rounding) identical to the sensitivities computed by the optimization procedure utilized by Schneider and Davison (2005). Table 3 also demonstrates that while changeover responses (LR and RL) were generally less preferred than repetition responses (LL and RR), as reported by Schneider and Davison, there was considerable variation across subjects as to the degree of bias. In particular, Subject 113 showed a clear preference for RR but not for LL, while Subject 115 treated LL and LR as approximately equal. Our analysis therefore enriches the general finding of a repetition bias, and provides the range of biases demonstrated by individual subjects and for different response units.

We applied the Barycentric Matching Model to other published data sets as well. For example, in Davison and McCarthy (1994), pigeons responded to a three-schedule concurrent contingency operated via a Findley switch procedure. In the high discriminability (HD) condition in which color cues were provided,

the pigeons readily distinguished between schedules, but in the low discriminability (LD) condition two of the cues were very similar. We found that sensitivities were much higher in the HD condition (mean = 0.851, SE = 0.085) than in the LD condition (mean = 0.532, SE = 0.088), without any substantial shifts in bias values. These results generally agree with those reported by Davison and McCarthy, who found lower pair-wise sensitivities in the LD condition than in the HD condition.

#### Future Directions

The Barycentric Matching Model is a flexible tool that extends the experimental scope of matching research. For example, it can be used to explore the relationship between reinforcer frequency and reinforcer magnitude, a relationship that has been studied by extending the Generalized Matching Law (e.g. Lau & Glimcher, 2005). In such an analysis, the model might take the following form:

$$\frac{bk_A \cdot (R_A)^{s_R} \cdot (M_A)^{s_M}}{B_A} = \frac{bk_B \cdot (R_B)^{s_R} \cdot (M_B)^{s_M}}{B_B} = \dots = \frac{bk_N \cdot (R_N)^{s_R} \cdot (M_N)^{s_M}}{B_N} \quad (8)$$

Here, while  $R_X$  indicates the number of reinforcers,  $M_X$  indicates the quantity delivered per reinforcing event, and instead of a single sensitivity  $s$ ,  $R_X$  and  $M_X$  each have individual sensitivities (here indicated as  $s_R$  and  $s_M$ , respectively). Similar extensions of the formula could be made for other variables, such as reinforcer delay or discriminability. If any of these variables result in a bias, the degree of bias can be quantified.

Schneider and Davison’s (2005) study suggests another possible direction: the study of response sequences. Although Schneider and

Davison looked at response dyads, they did so treating each pair as a discrete trial. There is no reason, however, why such an analysis could not be applied to traditional concurrent responding using a moving window. For example, the response sequence BAAB could be decomposed into single responses (B, A, A, B), dyads (BA, AA, AB), and triads (BAA, AAB). The chief obstacle to such an analysis would be the volume of data necessary. The resulting analysis could provide a bridge between traditional matching analyses (generally understood to be “molar” in nature) and sequential (or “molecular”) analyses of behavior.

Another potential application for the Barycentric Matching Model is to attempt to model real-world data. Because of its capacity for comparing many options simultaneously, the model might be modified to analyze (for example) the relationship between sale price and consumer consumption in a retail context where many different items are available; or patterns of Internet use (such as frequency of visit vs. frequency of update across many sites). While experimental confounds make working with such real-world data difficult, it is nevertheless a direction for behavior analysis with considerable potential gains, both in terms of raising new questions and in terms of generating new applications of existing theory.

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## APPENDIX A: THE SALT METHOD FOR COMPUTING BARYCENTRIC BIAS

The “SALT Method” for deriving the Barycentric Matching Model relies on a generalization of Equation 2 originally proposed by Natapoff (1970) to analyze matching on more than two operanda. SALT stands for “Symmetric Approximation by Leading Term,” and has been discussed in a number of publications (Aparicio & Cabrera, 2001; Schneider & Davison, 2005).

Instead of estimating bias from a series of pairwise comparisons, as is done in the

Pairwise Method, SALT compares an operandum to an amalgamation of all available operanda:

$$\frac{B_A}{\sqrt[N]{B_A \cdot B_B \cdot \dots \cdot B_N}} = k \left( \frac{R_A}{\sqrt[N]{R_A \cdot R_B \cdot \dots \cdot R_N}} \right)^s \quad (A1)$$

In Equation A1, responses ( $B$ ) and reinforcers ( $R$ ) from  $N$  operanda are included. The denominator of each fraction is simply the geometric mean of the relevant data from all operanda. Thus, the number of responses on operandum  $A$ , shown by  $B_A$ , is relative to the geometric mean of responses on all of the



operanda; and the same holds for the reinforcers. As in Equation 1, the equation also includes a bias parameter ( $k$ ) and a sensitivity parameter ( $s$ ).

In order to estimate the bias parameters associated with each operandum, the  $k$  parameter must be made more explicit (much as we did with Equation 2). Schneider & Davison (2005) proposed the following way to do this:

$$\frac{B_A}{\sqrt[N]{B_A \cdot B_B \cdot \dots \cdot B_N}} = \left( \frac{k_A}{\sqrt[N]{k_A \cdot k_B \cdot \dots \cdot k_N}} \right) \cdot \left( \frac{R_A}{\sqrt[N]{R_A \cdot R_B \cdot \dots \cdot R_N}} \right)^s \quad (\text{A2})$$

In this form, it becomes clear that SALT's  $k$  compares the bias parameter for one operandum to a composite of all bias parameters.

The SALT method for estimating barycentric bias requires a logarithmic transformation of Equation A2, analogous to that in Equation 5:

$$\log \left( \frac{B_A}{\sqrt[N]{B_A \cdot \dots \cdot B_N}} \right) = \log \left( \frac{k_A}{\sqrt[N]{k_A \cdot \dots \cdot k_N}} \right) + s \cdot \log \left( \frac{R_A}{\sqrt[N]{R_A \cdot \dots \cdot R_N}} \right) \quad (\text{A3})$$

Equation A3 puts the data in a form that permits the parameters to be estimated with a linear regression. Each intercept is an estimate of the bias parameter for that operandum. Thus, by repeating the regression with each operandum in the numerator, we calculate the five intercepts in order to estimate those five biases.

Note that the intercept parameters derived using this method appear to estimate the same values as the geometric means presented in the Pairwise Method. For example, for the Left Key, the intercept in Equation 8 allows us to estimate  $\frac{k_{LK}}{\sqrt[5]{k_{LK} \cdot k_{CK} \cdot k_{RK} \cdot k_{LL} \cdot k_{RL}}}$ . As a consequence, the final step of the SALT Method is the same as that of the Pairwise Method: compute the sum of the five intercepts and multiply each value by  $\text{Sum}/5$ . The result will be five  $bk_W$  parameters, scaled to  $T = 5$ .

In the case of Subject 1, the resulting barycentric bias parameters for (LK : CK : RK : LL : RL) were (0.736 : 0.725 : 0.747 : 1.461 : 1.331) respectively, results very close to those derived from the Pairwise Method.

## APPENDIX B: THE PAIRWISE METHOD AND THE SALT METHOD COMPARED

There is an important distinction between assumptions made by a model (such as the Generalized Matching Law) and assumptions made by a method for estimating model parameters (such as a linear regression). Arithmetically, the Pairwise and SALT Methods are identical: Not only does SALT (exemplified by Equation A2) reduce to the traditional Generalized Matching Law (exemplified by Equation 2), but it is more generally the case that both Equations 2 and A2 can be achieved through transformation of Equation 4. The difference between Pairwise and SALT methods is not, then, theoretical. Instead, the difference derives from the process of estimating model parameters using linear regressions on logarithmic coordinates. In other words, which method to choose is a practical question.

The key assumption in using a linear regression is that the spread of the values for the independent variable will be large enough to overcome the error (or "noise") in the dependent variable. If the experimental data are such that the regression lines for either method have a low  $R^2$  value, then the parameter estimates derived from those regressions will have a high likelihood of differing from the "true" values, assuming that many replications can arrive at those values.

The conditions under which the SALT Method is less effective than Pairwise are relatively straightforward. Imagine an experiment in which, in each session across all phases, subjects received very close to 20% of their reinforcers from the Left Lever. When Equation A3 is used to estimate the Left-Lever bias, there will be little to no spread along the  $x$ -axis of the regression, and the intercept will instead be determined chiefly by the spread of error along the  $y$ -axis. Thus, each operandum must receive a variety of different proportions of reinforcement over the course of the study for the SALT Method to be effective.

The Pairwise Method, by contrast, depends not on variations in a single operandum's proportion of reinforcement, but rather on variations in the *ratios* of reinforcers across pairs of operanda. Imagine an experiment in which Phase 1 delivered 40% of the reinforcers

to the Left Lever and 20% to the Right Lever, while Phase 2 saw 20% and 10% respectively. The ratio of (LL : RL) would be (2 : 1) in both cases, and the  $x$ -axis of that regression will show little spread. Thus, the proportion of reinforcement each operandum received must vary relative to other operandum for the Pairwise Method to be effective.

Because our experimental design had large variations in both per-operandum reinforcement frequencies and in pairwise reinforcement ratios, both methods for calculating barycentric bias were effectively equivalent. However, we can describe other circumstances where one method or the other is less effective. Below, we present three hypothetical scenarios of reinforcement across five response alternatives: in two of them, one of our methods is expected to face difficulties, while in the third, both methods will be effective. All scenarios involve an experiment with five operandum and two different phases, and all involve the same proportions of reinforcement.

In the first scenarios, the two phases involve the following proportions of reinforcement: Phase 1 = (0.08, 0.11, 0.17, 0.26, 0.38), Phase 2 = (0.08, 0.38, 0.26, 0.17, 0.11). Here, the same five proportions are arranged in two different ways. The SALT Method will be vulnerable to error in this case because Operandum 1 has received a proportion of 0.08 in both cases. However, the Pairwise method should remain effective in this case because all the pairwise ratios are distinct. For example, (A : B) is (0.08 : 0.11) in Phase 1 and is (0.08 : 0.38) in Phase 2, a difference of about a factor of 3.

In the second scenario, the two phases are as follows: Phase 1 = (0.08, 0.11, 0.17, 0.26, and 0.38), Phase 2 = (0.17, 0.26, 0.38, 0.08, 0.11). Here, SALT will be quite effective, because every operandum differs between the two phases. However, the Pairwise Method will

face difficulties because many of the paired ratios do not differ. For example, (B : C) is (0.11 : 0.17) in Phase 1, which is about a ratio of (1 : 1.5). In Phase 2, (B : C) is (0.26 : 0.38), which is also about a ratio of (1 : 1.5). Because of this, a regression using (B : C) will have little variance, and be prone to considerable error.

In the third scenario, Phase 1 = (0.08, 0.11, 0.17, 0.26, 0.38) and Phase 2 = (0.38, 0.17, 0.11, 0.26, 0.08). In this case, both the per-operandum proportions and the pairwise ratios differ between the two phases, so both the SALT and the Pairwise methods will be able to effectively estimate the Barycentric Matching Model.

The shortcomings exemplified by the above scenarios can largely be avoided through proper experimental design. A fully counterbalanced design (in which each operandum is at some point exposed to each proportion of reinforcement) will generally render both methods effective in generating parameter estimates. Even if a fully counterbalanced method is not implemented, a design that carefully avoids invariance of individual operandum (as in our first scenario) and pairwise invariance (as in our second scenario) allows both methods to be applied effectively. Further, because extant data may not be properly counterbalanced, we encourage the mindful selection of an analytic method. The ability to produce reasonable estimates from non-ideal data sets is vital to broadening the relevance and applicability of the Barycentric Matching Model, and matching analyses generally.

Thus, we wish to caution against the assumption that the two methods of analysis are interchangeable. We also want to emphasize that the distinctions between methods do not constitute a distinction between models. In all cases, our two analytic methods are estimating the same relationships, embodied by Equation 4.

APPENDIX C  
Barycentric Matching Model Parameters Per Subject.

Subject	$bk_{LeftKey}$	$bk_{CenterKey}$	$bk_{RightKey}$	$bk_{LeftLever}$	$bk_{RightLever}$	$s$
1	0.7380	0.72441	0.7490	1.4678	1.3207	0.5256
2	0.6919	0.5924	0.6820	1.6166	1.4171	0.4536
3	0.6913	0.9115	0.6428	1.5383	1.2162	0.4608
4	0.6245	0.6346	0.6565	1.8382	1.2461	0.4798
5	0.6940	0.8287	0.6626	1.8335	0.9813	0.5402
6	0.8901	0.8280	0.8315	1.3057	1.1447	0.4304
7	0.7109	0.6864	0.7185	1.6555	1.2287	0.4900
8	0.8342	0.7683	0.9262	1.3122	1.1591	0.5290
9	0.6981	0.6875	0.6262	1.6635	1.3246	0.3886
10	0.6917	0.76449	0.8618	1.5106	1.1714	0.4465
11	0.8108	0.8132	0.8441	1.2965	1.2354	0.4971
12	0.8571	0.7959	0.8522	1.3017	1.1932	0.5195
13	0.5836	0.7136	0.6841	1.6911	1.3275	0.6198
14	0.7098	0.7441	0.7227	1.4944	1.3289	0.5210
15	0.7122	0.7292	0.7231	1.4319	1.4036	0.5595
16	0.8886	0.5712	0.7426	1.6133	1.1843	0.5848
17	0.7283	0.7301	0.7907	1.2941	1.4568	0.4809
18	0.8276	0.7660	0.6557	1.6041	1.1466	0.3676
19	0.7437	0.6982	0.8122	1.2239	1.5219	0.4981
20	0.7511	0.6126	0.7065	1.6629	1.2669	0.4629